For some function f and some **non-zero** number a, the derivative of f at a is given by **SCORE: _____ / 20 PTS**

$$\lim_{h\to 0}\frac{2^{\csc[\pi(h+\frac{1}{2})]}-2}{h}$$

[a] Find f and a. Show that your answers are correct using the definition of the derivative at a point.

4 $f(x) = 2^{\csc \pi x}$, $a = \frac{1}{2}$ 4

2

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h} = \boxed{\lim_{h\to 0}\frac{2^{\csc\pi(\frac{1}{2}+h)}-2^{\overset{1}{\cancel{\csc\frac{\pi}{2}}}}}{h}}$$

$$= \lim_{h\to 0}\frac{2^{\csc\pi(h+\frac{1}{2})}-2}{h}$$

[b] Find the value of the limit, by evaluating $f'(a)$.

3

$$f'(x) = 2^{\csc\pi x}\cdot\ln 2\cdot -\csc\pi x\cot\pi x\cdot\pi$$ 2

3

$$= 2^{\csc\frac{\pi}{2}}\cdot\ln 2\cdot -\csc\frac{\pi}{2}\cot\frac{\pi}{2}\cdot\pi$$

$$= 2\cdot\ln 2\cdot -1\cdot 0\cdot\pi = 0$$ 2

Prove the derivative of $\tan x$ **using the definition of the derivative function**. **Show all steps.** SCORE: _____ / 15 PTS

Do NOT use the derivative shortcuts (such as the product rule etc.).

You may use the value of the two trigonometric limits proved in lecture without proving them again.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \lim_{h \to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \quad 4$$

$$= \lim_{h \to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \quad 4$$

$$= \lim_{h \to 0} \frac{\sin(\cancel{x}+h-\cancel{x})}{h} \cdot \lim_{h \to 0} \frac{1}{\cos(x+h)\cos x} \quad 5$$

$$= 1 \cdot \frac{1}{\cos^2 x} = \sec^2 x \quad 2$$

If g is a function and $f(x) = x^2 g(\frac{1}{x})$, find a formula for $f''(x)$, which may involve g, g' and/or g''. **SCORE: _____ / 20 PTS**

$$= x^2 g(x^{-1})$$

$$f'(x) = 2xg(x^{-1}) + x^2 g'(x^{-1})(-x^{-2})$$

$$= \underbrace{2xg(x^{-1})}_{3} \underbrace{- g'(x^{-1})}_{5}$$

$$f''(x) = 2g(x^{-1}) + 2xg'(x^{-1})(-x^{-2}) - g''(x^{-1})(-x^{-2})$$

$$= \underbrace{2g(\tfrac{1}{x})}_{2} \underbrace{- \tfrac{2}{x} g'(\tfrac{1}{x})}_{5} \underbrace{+ \tfrac{1}{x^2} g''(\tfrac{1}{x})}_{5}$$

Prove that $y = mx + b$ and $x^2 + y^2 - 2by = c$ are orthogonal trajectories. **SCORE: _____ / 20 PTS**

$$\underbrace{\frac{dy}{dx} = m}_{1} \qquad \underbrace{2x + 2y\frac{dy}{dx} - 2b\frac{dy}{dx} = 0}_{6}$$

$$\underbrace{\frac{dy}{dx} = \frac{2x}{2b-2y} = \frac{x}{b-y}}_{3}$$

$$m \cdot \frac{x}{b-y} = \underbrace{\boxed{\frac{mx}{b-y}}}_{4} = \frac{\overbrace{y-b}^{3}}{b-y} = \underbrace{-1}_{3}$$

Let $f(x) = (1+\ln x)^{\ln x}$. $\quad f(e) = (1+1)^1 = 2$ **SCORE: _____ / 25 PTS**

[a] If x changes from e to 3, find dy.

3 $\ln y = \ln x \ln(1+\ln x)$

3 $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(1+\ln x) + \ln x \frac{1}{1+\ln x}\frac{1}{x}$ 5

3

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\ln(1+\ln x) + \frac{\ln x}{1+\ln x}\right)$

$\frac{1}{2}\frac{dy}{dx}\Big|_{x=e} = \frac{1}{e}\left(\ln(1+1) + \frac{1}{1+1}\right) = \frac{1}{e}\left(\ln 2 + \frac{1}{2}\right)$ 3

$\frac{dy}{dx}\Big|_{x=e} = \frac{2}{e}\left(\ln 2 + \frac{1}{2}\right)$ 2

$dy = \frac{2}{e}\left(\ln 2 + \frac{1}{2}\right)\Delta x$

$= \frac{2}{e}\left(\ln 2 + \frac{1}{2}\right)(3-e)$ 3

[b] Approximate $f(3)$ **using your answer to part [a]**.

$f(3) \approx f(e) + dy$

$= 2 + \frac{2}{e}\left(\ln 2 + \frac{1}{2}\right)(3-e)$ 3

Find $\frac{d^3}{dx^3}\arctan\frac{1}{x^2}$. **SCORE: _____ / 25 PTS**

HINT: Simplify often.

$$\frac{d}{dx}\arctan\frac{1}{x^2} = \boxed{\frac{1}{1+\left(\frac{1}{x^2}\right)^2}} \cdot \boxed{-\frac{2}{x^3}} = -\frac{2}{x^3+\frac{1}{x}} = \boxed{-\frac{2x}{x^4+1}}$$

3 3 2

$$\frac{d^2}{dx^2}\arctan\frac{1}{x^2} = -\frac{2(x^4+1)-2x(4x^3)}{(x^4+1)^2} = \frac{6x^4-2}{(x^4+1)^2}$$

4 2 2

$$\frac{d^3}{dx^3}\arctan\frac{1}{x^2} = \frac{24x^3(x^4+1)^{\cancel{2}}-(6x^4-2)2\cancel{(x^4+1)}4x^3}{(x^4+1)^{\cancel{4}\,3}}$$

4 2

$$= \frac{8x^3(3x^4+3-6x^4+2)}{(x^4+1)^3}$$

$$= \boxed{\frac{8x^3(5-3x^4)}{(x^4+1)^3}}$$

3

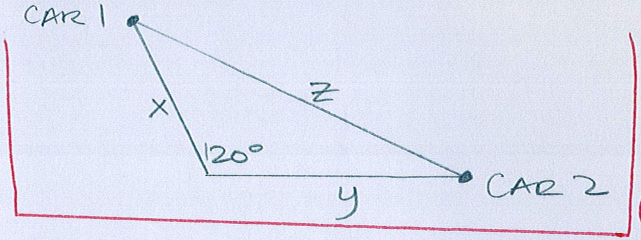
Two roads meet at an intersection at a 120° angle.

SCORE: _____ / 25 PTS

On one road, a car is driving towards the intersection at 60 miles per hour.
On the other road, a car is driving away from the intersection at 80 miles per hour.
At the moment when the first car is 4 miles from the intersection, and the second car is 3 mile from the intersection, are the cars getting closer or farther apart (as measured by the direct distance between them), and how quickly ?

You must state/show clearly what each variable you use represents.

You must show the units during the intermediate steps of your work, and you must state the units for the final answer.



3

$$\frac{dx}{dt} = -60\,\frac{mi}{h} \qquad \frac{dy}{dt} = 80\,\frac{mi}{h}$$

WANT $\left.\dfrac{dz}{dt}\right|_{x=4\,mi,\ y=3\,mi}$

4 $z^2 = x^2 + y^2 - 2xy\cos 120^\circ = x^2 + y^2 + xy$

$$z^2 = (4\,mi)^2 + (3\,mi)^2 + (4\,mi)(3\,mi) = 37\,mi^2$$

$$z = \sqrt{37}\,mi$$

4 $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + \dfrac{dx}{dt}y + x\dfrac{dy}{dt}$

$$2\sqrt{37}\,mi\,\frac{dz}{dt} = 2(4\,mi)\left(-60\,\frac{mi}{h}\right) + 2(3\,mi)\left(80\,\frac{mi}{h}\right) + \left(-60\,\frac{mi}{h}\right)(3\,mi) + (4\,mi)\left(80\,\frac{mi}{h}\right)$$

7

$$2\sqrt{37}\,\frac{dz}{dt} = 140\,\frac{mi}{h}$$

$$\frac{dz}{dt} = \frac{70}{\sqrt{37}}\,\frac{mi}{h}$$

2

THE CARS ARE GETTING FARTHER APART BY $\boxed{\dfrac{70}{\sqrt{37}}}$ MILES PER HOUR ← MUST BE SHOWN IN WORK

3 2